***On the Characters Table of Finite Groups***

 *This lecture is presenting an introduction to the character tables of abelain groups,The irreducible characters of the finite group G are class functions , and the number of them is equal to the number of conjugacy classes of G.*

 *It is, therefore , convenient to record all the values of all the irreducible characters of G in a square matrix . This matrix is called the characters table of G .*

*For more information , see [2] .*

***Definition (1):[2]***

 Let χ,χ,…,χbe the irreducible characters of the finite group G and let g1,g2,…,gbe representatives of the conjugacy classes of G .

The matrix whose *ij*-entry is χ*i*( g*j*) (for all *i* , *j* with 1< *i* , *j* < *k* ) is called ***character table*** of G which is denoted by ≡(G).

 It is usual to number the irreducible characters and conjugacy classes of G, (CL), () so that χ(g)=1,for all *g*$\in $ G, the trivial character; beyond this numbering is arbitrary . This could be noticed in the characters table , (or,in practice , by conjugacy class representatives ).

 Such that g∈CL denoting the number elements in CLby *h*, we have the class equation ++…+=│G│and the degree of *k* distinct representations of G over C by *ni* ; i =1,2,…,*k* . Besides, the size of the centralizer is C /= m.

 The following table presents a typical characters table, the × square matrix whose rows are indexed by the irreducible characters of G and the columns are indexed by the conjugacy classes.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  CL |  CL |  CL |  **…** |  CL |  **…** |  CL |
|   |   |   |  … |   |  … |   |
|  *χ* |  *1* |  *1* |  ***…*** |  *1* |  ***…*** |  *1* |
|  *χ* |  *n* |  *χ* |  ***…*** |  *χ* |  ***…*** |  *χ* |
|  *χ* |  *n* |  *χ* |  ***…*** |  *χ* |  ***…*** |  *χ* |
|  |  |  |  |  |  |  |
|  *χ* |  *n* |  *χ(g)* |  ***…*** |  *χ(g)* |  ***…*** |  *χ* |

Table (1)

***Lemma (2):[2]***

 The characters table of G is an invertible matrix .

***Example (3):***

 Consider the dihedral group D3 . It has three conjugate classes

[(I)]={( I)},

[]={ ,} ,

[]={,,}

and it has three non-equivalent irreducible representations,

 is the principal representation i.e. (g)= [1] , gD3 .



For all gD3 .

and  is defined as follows :



(

where .

 If *χ1*, *χ2*and *χ3* are the characters of  respectively,

then we have *χ1*(g)=1 , gD3,

*χ2*( I)=*χ2*(*r*)= *χ2*(*r2*)= 1 ,

*χ2*(*S*)=*χ2*(*S r*)= *χ2*(*S r2*)= -1 ,

*χ3*( I )=2 , *χ3*(*r*)= *χ3*(*r2*)= -1 ,

*χ3*(*S*)=*χ3*(*S r*)= *χ3*(*S r2*)= 0 ,

 Since all the elements in the conjugacy classes have the same character , thus they are equivalent , then we can write table (2), the characters table of D3 , as follows :

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  | [I] | [] | [] |
|  |  | 1 | 2 | 3 |
|  **≡**D3 = | *χ1* | 1 | 1 | 1 |
|  | *χ2* | 1 |  1 |  -1 |
|  | *χ3* | 2 | -1 | 0 |

 Table (2)

***Theorem (4):[2]***

 Let  be all the irreducible characters of G and let

 be the representatives of the conjugacy classes of G. Then the following relations hold for any 1< *i* , *j* < *k* :

1. The row orthogonality relations :



1. The column orthogonality relations :

 where  .

***Characters Table of Finite Abelian Groups(5):[3]***

 The problem of constructing the characters table can be solved easily when G is a finite abelain group .

Each element forms a conjugacy class by itself so that

 k =  and  ,

 therefore χ(1) =1, for all i .

 Hence, the matrix representations are identified with their characters i.e.  .

 The elements of G can be listed in the first row and put, , 1≤ i ≤ *n* , 1≤ j ≤ *n*-1 . If G=Cn=< *r*> is the ***cyclic group*** of the order *n* generated by *r* , let  = ebe a primitive n-th root of unity, then ***the general form* *of the* *characters table of the cyclic group Cn*** is:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  CL |  I  |  *r* |  *r* |  **…** |  *r* |
|  |   |  1 |  1 |  1 |  **…** |  1 |
| **≡(Cn)=** |  *χ* |  *1* |  *1* |  *1* |  ***…*** |  *1* |
|  |  *χ* |  *1* |  | *2* |  ***…*** | *n-1* |
|  |  *χ* |  *1* | *2* | *4* |  ***…*** | *n-2* |
|  |  |  |  |  |  |  |
|  |  *χ* |  *1* | *n-1* | *n-2* |  ***…*** |  |

 Table (3)

***Example (6):***

The characters table of the cyclic group C6=is given by :

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| ***CLα*** | ***1*** | ***r*** | ***r2*** | ***r3*** | ***r4*** | ***r5*** |
| ***| CLα|*** | ***1*** | ***1*** | ***1*** | ***1*** | ***1*** | ***1*** |
| ***χ1*** | *1* | *1* | *1* | *1* | *1* | *1* |
| ***χ2*** | *1* |  | ***2*** | ***3*** | ***4*** | ***5*** |
| ***χ3*** | *1* | ***2*** | ***4*** | ***1*** | ***2*** | ***4*** |
| ***χ4*** | *1* | ***3*** | ***1*** | ***3*** | ***1*** |  |
| ***χ5*** | *1* | ***4*** | ***2*** | ***1*** | ***4*** | ***2*** |
| ***χ 6*** | *1* | ***5*** | ***4*** | ***3*** | ***2*** |  |

 Table (4)

where=e .

***Theorem (7):[1]***

 Let T: G1→GL(*n*,F) and T: G2→GL(*m*,F) are two irreducible representations of the groups G1 and G2 with characters χand χ respectively ,

 then TT is irreducible representation of the group G1G2 with the character χχ .

***Example (8):***

 The characters table of *C2* = is :

|  |  |  |
| --- | --- | --- |
| **CLα** | **1** | **r\*** |
| **| CLα|** | **1** | **1** |
|  | 1 | 1 |
|  | 1 | -1 |

 ≡ *C2=*

 Table(5)

The characters table of *C5* =is :

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **CLα** | **1** | **r** | **r2** | **r3** | **r4** |
| **| CLα|** | **1** | **1** | **1** | **1** | **1** |
| ***χ1*** | 1 | 1 | 1 | 1 | 1 |
| ***χ 2*** | 1 |  | *2* | *3* |  *4* |
| ***χ 3*** | 1 | *2* | *4* |  |  *3* |
| ***χ 4*** | 1 | *3* |  | *4* |  *2* |
| ***χ 5*** | 1 | *4* | *3* | *2* |  |

*≡ C5*=

 Table(6)

Then the irreducible character table *C2C5* is :

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **CLα** | **[1,1]** | **[1, r]** | **[1,r2]** | **[1,r3]** | **[1,r4]** | **[r\*, 1]** | **[ r\*, r]** | **[r\*, r2]** | **[r\*, r3]** | **[r\*, r4]** |
| **| CLα|** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** |
| ***χ11*** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** |
| ***χ12*** | **1** |  |  **2** |  **3** |  **4** | **1** |  |  **2** |  **3** |  **4** |
| ***χ13*** | **1** |  **2** |  **4** |  |  **3** | **1** |  **2** |  **4** |  |  **3** |
| ***χ14*** | **1** |  **3** |  |  **4** |  **2** | **1** |  **3** |  |  **4** |  **2** |
| ***χ15*** | **1** |  **4** | **3** |  **2** |  | **1** |  **4** |  **3** |  **2** |  |
| ***χ21*** | **1** | **1** | **1** | **1** | **1** | **-1** | **-1** | **-1** | **-1** | **-1** |
| ***χ22*** | **1** |  |  **2** |  **3** |  **4** | **-1** | **-** | **-** **2** | -**3** | - **4** |
| ***χ23*** | **1** | **2** |  **4** |  |  **3** | **-1** | **-** **2** | **-** **4** | **-** | **-** **3** |
| ***χ24*** | **1** |  **3** |  |  **4** |  **2** | **-1** | **-** **3** | **-** | **-** **4** | **-** **2** |
| ***χ25*** | **1** |  **4** |  **3** |  **2** |  | **-1** | **-** **4** | **-** **3** | **-****2** | **-** |

Table(7)

**References**

 [1] C.W. Curits & I. Renier " Representation Theory of Finite Groups and Associative Algebra " ,AMS Chelsea publishing ,1962 , printed by the AMS ,2006 .

[2] G. D. James & M. Liebeck " Representations and groups " , 2nd

 edition ,Cambridge University press ,2001 .

[3] H.H. Abass," On The Factor Group of Class Functions Over The

 Group of Generalized Characters of D", M.Sc thesis, Technology

 University,1994.