CONGRUENCES ON INVERSE SEMIGROUPS

Inverse elements in semigroups

We say that $y \in S$ is regular if there exists $x \in S$ such that $y = yxy$.

We say that $y \in S$ is an **inverse element** of $x \in S$, if
$x = xyx$ and $y = yxy$.
Note that an inverse element of x, if such an element exists, need not be unique.

Lemma 1. *Each regular element $x \in S$ has an inverse element.*

Proof. If $x \in S$ is regular, then for some $y \in S$, $x = xyx$. Now, $yxy = yxy \cdot x \cdot yxy$,

and so yxy is also regular. Also, $x = x \cdot yxy \cdot x$ and consequently yxy is an inverse
element of x.

Theorem 2. *Let ρ be a congruence of a regular semigroup S, then*
$x\rho \in E_{S/\rho} \Longrightarrow \exists e \in E_S : x\rho = e\rho$.

Theorem 3. *If $\alpha : S \to P$ is a homomorphism from a regular semigroup S, then $\alpha(S)$
is regular. In particular, if α is an epimorphism, then P is regular.*

A semigroup S is called an **inverse semigroup**, if each $x \in S$ has a *unique* inverse
element x^{-1}:

$x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$.

The semilattice of idempotents

If $e \in E_S$ for an inverse semigroup S, then $eee = e$, and hence for all idempotents e,

$e^{-1} = e$.

Theorem 4. *Let* S *be an inverse semigroup. Then the idempotents* E_S *form a subsemigroup of* S*. Moreover,* E_S *is a semilattice, that is, the idempotents of an inverse semigroup commute.*

Proof. Let $e, f \in E_S$ and consider the (unique) inverse element $x = (ef)^{-1}$ of ef.
Now,

$$ef = ef \cdot x \cdot ef = ef \cdot xe \cdot ef \text{ and}$$

$$ef = ef \cdot x \cdot ef = ef \cdot fx \cdot ef$$

and

$$xe \cdot ef \cdot xe = xefx \cdot e = xe ,$$

$$fx \cdot ef \cdot fx = f \cdot xefx = fx .$$

This means that $x = (ef)^{-1} = xe = fx$. Here $x \in E_S$, since

$$x^2 = xe \cdot fx = x \cdot ef \cdot x = x ,$$

and so $ef \in E_S$ for all $e, f \in E_S$, that is, E_S is a subsemigroup of S.

Further, E_S is commutative: For $e, f \in E_S$, also $ef, fe \in E_S$, and

$ef \cdot fe \cdot ef = efef = (ef)^2 = ef$ and $fe \cdot ef \cdot fe = fefe = (fe)^2 = fe$,

meaning that $fe = (ef)^{-1} = ef$.

Corollary 5. *Assume* $S = [X]_S$*. If each generator* $x \in X$ *has a unique inverse element,*
then S *is an inverse semigroup:*

$$(x_1 x_2 \ldots x_n)^{-1} = x_n^{-1} x_{n-1}^{-1} \ldots x_1^{-1}$$

for all $x_i \in X$.

Corollary 6. *In an inverse semigroup* S*, for all* $x \in S$*,* $x = (x^{-1})^{-1}$.

A characterization

Theorem 7. *Let* S *be a semigroup. The following are equivalent:*
1. S *is an inverse semigroup.*

2. S *is regular and its idempotents commute.*
3. Each L*-class and* R*-class contains an idempotent.*
Where L and R are Green-relations.
Proof. Case (1) implies Case (2) by Theorem 4.
Suppose Case (2). Since each L-class and R-class contains a unique idempotent.
For the uniqueness let $f \in L_e$, where $e, f \in E_S$. Hence eLf, and therefore there are $x, y \in S^1$ such that $e = xf$ and $f = ye$. From here we obtain
$e = xf = xff = ef = fe = yee = ye = f$.
Similarly, eRf implies that $e = f$. So Case (2) implies Case (3).
Suppose Case (3). Now each D-class contains an idempotent, where D = L°R, and hence, by (D-*class is regular if and only if it contains an idempotent*), each $x \in S$ has an inverse element. Suppose an element x has two inverse elements y and z. Now, $yx, zx \in E_S$ with $yxLx$ and $zxLx$. Then, by assumption, $yx = zx$.
A similar reasoning using R shows that $xy = xz$. Therefore $y = yxy = zxz = z$, and Case (1) follows.
Corollary 8. *Let* S *be an inverse semigroup. Then*
$\forall x \in S : x^{-1}E_Sx \subseteq E_S$.
Theorem 9. *Let* S *be an inverse semigroup, and let* $x, y \in S$ *and* $e, f \in E_S$. *Then*
1. $xLy \iff x^{-1}x = y^{-1}y$.
2. $xRy \iff xx^{-1} = yy^{-1}$.
3. $eDf \iff \exists z \in S : e = zz^{-1}$ *and* $f = z^{-1}z$.

Partial ordering inverse semigroups

Recall that in any semigroup S the idempotents can be partially ordered by the relation:
$e \leq f \iff ef = e = fe$.
This partial order generalizes in an inverse semigroup S to all elements of S as follows,
$x \leq y \iff \exists e \in E_S : x = ey$.

Indeed, here ≤ is

• reflexive, since $x = (xx^{-1}) \cdot x$, where $xx^{-1} \in E_S$;

• antisymmetric, since if $x = ey$ and $y = fx$, then $x = ey = eey = ex$, and so
$x = ey = efx = fex = fx = y$;

• transitive, since if $x = ey$ and $y = fz$, then also $x = ey = efz$, where $ef \in E_S$.

If you restrict ≤ onto E_S you get the above partial order of idempotents. Indeed, if
$e \leq f$, then there exists $g \in E_S$ such that $e = gf$, and here $e = gff = ef = fe$ as required.

Lemma 10. *In an inverse semigroup* S *we have*
$x \leq y \iff \exists e \in E_S : x = ye \iff xx^{-1} = yx^{-1} \iff x = xy^{-1}x$
$\iff xx^{-1} = xy^{-1} \iff x^{-1}x = y^{-1}x \iff x^{-1}x = x^{-1}y \iff x = xx^{-1}y$.

Partial mappings

Let $X \neq \varnothing$ be a set. A **partial mapping** $\alpha : X \to X$ is a function from a subset $Y = \text{dom}(\alpha)$ of X onto $\text{ran}(\alpha) = \alpha(Y) \subseteq X$. A partial mapping $\alpha : X \to X$ is **undefined** on all x not belonging to $\text{dom}(\alpha)$.

We say that a partial mapping $\alpha : X \to X$ is **injective**, if $\alpha(x) \neq \alpha(y)$ for all $x \neq y$ with $x, y \in \text{dom}(\alpha)$. The injective partial mappings form a semigroup, denoted I_X, under the usual composition:
$(\beta\alpha)(x) = \beta(\alpha(x))$ if $x \in \text{dom}(\alpha)$ and $\alpha(x) \in \text{dom}(\beta)$.
We observe that
$\text{dom}(\beta\alpha) = \alpha^{-1}(\text{ran}(\alpha) \cap \text{dom}(\beta))$ and
$\text{ran}(\beta\alpha) = \beta(\text{ran}(\alpha) \cap \text{dom}(\beta))$.

We denote by $\iota_Y : X \to X$ the partial function such that $\text{dom}(\iota_Y) = Y = \text{ran}(\iota)$

and $\iota_Y(y) = y$ for all $y \in Y$.

Theorem 11. I_X *is an inverse semigroup.*

The Vagner-Preston representation

Theorem 12. *Each inverse semigroup* S *has a faithful representation as a semigroup of injective partial mappings, that is, there exists an embedding* $\phi: S \to I_X$ *for some set* X.

Congruences of Inverse semigroups

Heritage of images

Lemma 13. *Let* S *be an inverse semigroup and* $\alpha: S \to P$ *a homomorphism. Then*
$\alpha(S)$ *is an inverse subsemigroup of* P.

Corollary 14. *If* ρ *is a congruence of an inverse semigroup* S*, then* S/ρ *is an inverse semigroup.*
Therefore,
Lemma 15. *Let* S *be an inverse semigroup, and* ρ *its congruence. Then*
$x \rho y \iff x^{-1} \rho y^{-1}$.

We obtain also that for each homomorphism $\alpha: S \to P$ for an inverse semigroup S,
$\forall x \in S : \alpha(x^{-1}) = \alpha(x)^{-1}$.

A subsemigroup T of an inverse semigroup S is called a **inverse subsemigroup**, if for all $x \in T$ also $x^{-1} \in T$, where x^{-1} is the inverse element of x in S. Notice that not all subsemigroups of an inverse semigroup are inverse subsemigroups.
The following lemma is an exercise.
Lemma 16. *Let* S *be an inverse semigroup, and let* A *be a subsemigroup of* S*. Then*
A *is an inverse subsemigroup of* S *if and only if* $x^{-1} \in A$ *for all* $x \in A$.

Lemma 17. *Let* S *be an inverse semigroup,* $\alpha : S \to P$ *an epimorphism, and let* $e \in E_P$ *. Then* $\alpha^{-1}(e)$ *is an inverse subsemigroup of* S*.*

Theorem 18. *Let* I *be an ideal of a semigroup* S*. Then* S *is an inverse semigroup if and only if* I *and* S/I *are inverse semigroups*

Kernels and traces
RES?

Let ρ be a congruence of a semigroup S.We define its **kernel** ker(ρ) and **trace** tr(ρ) as follows:

$\ker(\rho) = \{x \in S \mid x \rho e \text{ for some } e \in E_S\} = \cup\ e\rho \ \text{ for all } e \in E_S$

$\mathrm{tr}(\rho) = \rho\,(\mathrm{res})E = \{(e, f) \mid e, f \in E_S\}$.

Theorem 19. *Let* S *be an inverse semigroup. Then for all congruences* ρ *and* δ *,*

$\rho \subseteq \delta \iff \forall e \in E_S : e\rho \subseteq e\delta$.

Corollary 20. *For an inverse semigroup* S*,*

$\rho = \delta \iff \forall e \in E_S : e\rho = e\delta$

for all congruences ρ *and* δ *.*

We have then **Vagner's theorem**:

Theorem 21. *Let* S *be an inverse semigroup, and let* ρ *and* δ *be its congruences. Then*

$\rho = \delta \iff \ker(\rho) = \ker(\delta)$ *and* $\mathrm{tr}(\rho) = \mathrm{tr}(\delta)$.

In other words, If $\alpha : S \to P$ and $\beta : S \to T$ are epimorphisms from an inverse semigroup S, then ker(α) = ker(β) if and only if for all $x \in S$ and $e, f \in E_S$,

$\alpha(x) \in E_P \iff \beta(x) \in E_T$,

$\alpha(e) = \alpha(f) \iff \beta(e) = \beta(f)$.

Classifications according to traces

Congruences of an inverse semigroup are classified according to their traces.

Lemma 22. *For all* $x \in S$ *and* $e \in E_S$, $x^{-1}ex \in E_S$.

For a congruence ρ of an inverse semigroup S, we obtain a congruence ρ_{min} by defining

$x \rho_{min} y \iff \exists e \in E_S : xe = ye, \ x^{-1}x \rho e$ and $y^{-1}y \rho e$.

The next theorem states that ρ_{min} identifies as few elements as possible under the restriction that it should identify exactly the same idempotents as the original ρ . In this way the quotient S/ρ_{min} is as large as possible.

Theorem 23. *For a congruence* ρ *of an inverse semigroup* S*,* ρ_{min} *is the smallest congruence whose trace equals* tr(ρ).

In particular, we have that $\rho_{min} \subseteq \rho$ for all congruences ρ of an inverse semigroup S.

For a congruence ρ of an inverse semigroup S define ρ_{max} by

$x \rho_{max} y \iff \forall e \in E_S : x^{-1}ex \rho y^{-1}ey$.

Theorem 24. *Let* S *be an inverse semigroup and* ρ *its congruence. Then* ρ_{max} *is the largest congruence of* S *whose trace equals* tr(ρ).

The above theorem states that ρ_{max} identifies as many elements of S as possible with the restriction that it does not identify any idempotents unless ρ does so. Certainly,

$\rho \subseteq \rho_{max}$, and so the quotient S/ρ_{max} is an epimorphic image of S/ρ .

Group congruences

We say that a congruence ρ of a semigroup S is a **group congruence**, if S/ρ is a group.

The following lemma holds already for regular semigroups.

Lemma 25. *An inverse semigroup is a group if and only if it has a unique idempotent.*

For a congruence ρ of an inverse semigroup, we have by Theorem 2 that

$x\rho \in E_S/\rho \implies \exists e \in E_S : e\rho = x\rho$,

and hence

Theorem 26. *A congruence* ρ *of an inverse semigroup* S *is a group congruence if and only if* $tr(\rho) = E_S \times E_S$.

Proof. If $tr(\rho) = E_S \times E_S$, then S/ρ has exactly one idempotent by Theorem 2. In the other direction the claim is equally clear.
If ρ is a group congruence of an inverse semigroup S, then so is ρ_{min}, because now
$tr(\rho) = tr(\rho_{min}) = E_S \times E_S$. In particular, ρ_{min} is the smallest group congruence of S,
and for all group congruences δ of S, $\delta_{min} = \rho_{min}$.
The **smallest group congruence** of an inverse semigroup S is denoted by σ_S.
Let then ρ be a group congruence. Then $\sigma_S = \rho_{min} \subseteq \rho$.
Every group G, which is a homomorphic image of S, is a homomorphic image of the group S/σ_S, and in this sense S/σ_S is a maximal homomorphic image of S.
Remark 27. If S is *not* an inverse semigroup, it need not have the smallest group congruence.
As an example consider $(N_+, +)$. The group congruences of this semigroup are exactly $\rho_n = \{(p, q) \mid p \equiv p(\bmod n)\}$.
Theorem 28 (Munn). *In an inverse semigroup* S,
$x \sigma_S y \iff \exists e \in E_S : xe = ye$.

Theorem 29. *In an inverse semigroup* S,
1. $x \sigma_S y \iff \exists e \in E_S : ex = ey$.
2. $ker(\sigma_S) = \{x \in S \mid \exists y \in S : xy = x\}$.

Idempotent separating congruences

A group congruence puts all idempotents in the same congruence class. An idempotent separating congruence does the opposite, it puts different idempotents to different classes.
A congruence ρ of a semigroup S is **idempotent separating**, if
$\forall e, f \in E_S : e \rho f \implies e = f$.

From this definition we have immediately

Lemma 30. *If* ρ *is an idempotent separating congruence of an inverse semigroup* S, *then* $tr(\rho) = \iota_E = \{(e, e) \mid e \in E_S\}$.

By Theorem 24, for each inverse semigroup S there exists the **greatest idempotent separating congruence**, which will be denoted by μ_S.

Theorem 31. *For all inverse semigroups* S,

$$x \mu_S y \iff \exists e \in E_S : x^{-1}ex = y^{-1}ey .$$